The Buffon Needle Problem . . .

George Louis Leclerc, Comte de Buffon (born on 7 September 1707 at Montbord, Burgundy; died on 15 April 1788 in Paris) has 25 index entries in the *Encyclopaedia Britannica* (35-volume 9th & 10th Editions, 1875–1903). These cover topics such as Bees; Bionomics; the Classification of Monsters; Crystallisation; his influence on Darwin; Evolution; Humming-birds; the Mirrors of Archimedes; Natural History*; Ornithology; Parrots; Phalanger (a genus of Possums, which he named); Probability; his work on Species; and the Trumpeter-bird.

*His magnum opus, *Histoire Naturelle, générale et particulidère*, came out over decades in many volumes as "a general encyclopedia of the sciences".

Originally destined for Law, which he studied at the College of Jesuits in Dijon, he turned towards the physical sciences, especially Mathematics (for which he was gifted as a youth). He translated Newton's *Fluxions*, and Hales's *Vegetable Staticks***. He worked widely in Mathematics, Physics, Agriculture, and biological sciences.

**Stephen Hales, FRS, DD (17/09/1677 – 4/01/1761) was a clergyman who worked in many areas of science. He wrote, amongst other works, two volumes of *Statical Essays*. The first, *Vegetable Staticks*, describes experiments in plant physiology and biochemistry, and the second, *Haemastaticks*, experiments in animal physiology, including the study of blood pressure.

Buffon also has 9 entries in the index of Isaac Todhunter's *A History of the Mathematical Theory of Probability from the Time of Pascal to That of Laplace* (Cambridge, 1865; Textually unaltered reprint: New York, 1949 and 1965). These refer to: solutions of some problems in chances in his *Essai d'Arithmétique Morale*; his association with D'Alembert in his thoughts about experimental determination of probability and Buffon's tables of duration of life; a whole section (7 pages) on Buffon's mathematical work in Probability (the *Essai* appeared in 1777 in the fourth volume of his *Supplément à l'Histoire Naturelle* where it occupies 103 quarto pages, and is believed to have been composed in 1760); pertinence to the work of Condorcet; some of his solutions which were "borrowed" in Bicquilley's *Du Calcul des Probabilités. Par C. F. de Bicquilley, Garde-du-Corps du Roi*.

In the 23rd section of the *Essai d'Arithmétique Morale* (according to Todhunter) appear some problems involving Probability and Geometry.

In the first of these:

> "Suppose a large plane area be divided into equal regular figures, namely squares, equilateral triangles, or regular hexagons. Let a round coin be thrown down at random; required the chance that it shall fall clear of the bounding lines of the figures, or fall on one of them, or on two of them, and so on.
>
> These examples only need simple mensuration, and we need not delay on them; we have not verified Buffon's results."
> (Todhunter, 1965 reprint, page 346, section 649)

And then:

> "Buffon then proceeds to a more difficult example which requires the aid of the Integral Calculus. A large plane area is ruled with equidistant parallel straight lines; a slender rod is thrown down: required the probability that the rod will fall across a line. Buffon solves this correctly.
> [· · · *Buffon next describes a similar problem with a grid of lines*]
> Laplace, without any reference to Buffon, gives the problem in his *Théorie ... des Prob.*, pages 359–362." [Published 1812]
> (Todhunter, 1965 reprint, page 347, section 650)

Thus, finally (though with perhaps uncertain date) we are led to the original **Buffon Needle Problem**. Todhunter does not discuss any details of Buffon's solution, nor of Laplace's.

On the next page is a transcript from pages 359–360 of Laplace's *Théorie Analytique des Probabilités* (1812), as read from the website of the Bibliothèque Nationale de France. URL (all one line):

http://gallica.bnf.fr/Search?ArianeWireIndex=index&p=1&lang=EN&f_typedoc=livre&q=theorie+analytique+des+probabilites&x=0&y=0

From *Théorie Analytique des Probabilités*, pp. 359–360

"Enfin, on pourrait faire usage du calcul des probabilités, pour rectifier les courbes ou carrer leurs surfaces. Sans doute, les géomètres n'emploiront pas ce moyen; mais comme il me donne lieu de parler d'un genre particulier de combinaisons du hasard, je vais exposer en peu de mots.

Imaginons un plan divisé par des lignes parallèles, équidistantes de la quantité a; concevons de plus un cylindre très-étroit dont $2r$ soit la longueur, supposée égale ou moindre que a. On demande la probabilité qu'en le projetant, il rencontrera une des divisions du plan.

Elevons sur un point quelquonque d'une des ces divisions, une perpendiculaire prolongée jusqu'à la division suivante. Supposons que le centre du cylindre soit sur cette perpendiculaire, et à la hauteur y au-dessus de la première de ces deux divisions. En faisant tourner le cylindre autour de son centre, et nommant ϕ l'angle que le cylindre fait avec la perpendiculaire, au moment où il rencontre cette division; 2ϕ sera la partie de la circonférence décrite par chaque extrémité du cylindre, dans laquelle il rencontre la division; la somme de toutes ces parties sera donc $4\int \phi\, dy$, ou $4\phi y - 4\int y\, d\phi$; or on a $y = r.\cos\phi$; cette somme est donc

$$4\phi y - 4r.\sin\phi + \text{constante}$$

Pour déterminer cette constante, nous observerons que l'intégrale doit s'étendre depuis y nul jusqu'à $y = r$, et par

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conséquent depuis $\phi = \frac{\pi}{2}$ jusquà $\phi = 0$, ce qui donne

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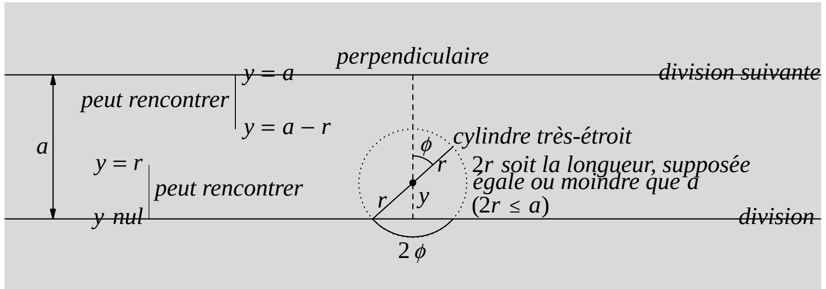
$$\text{constante} = 4r\,;$$

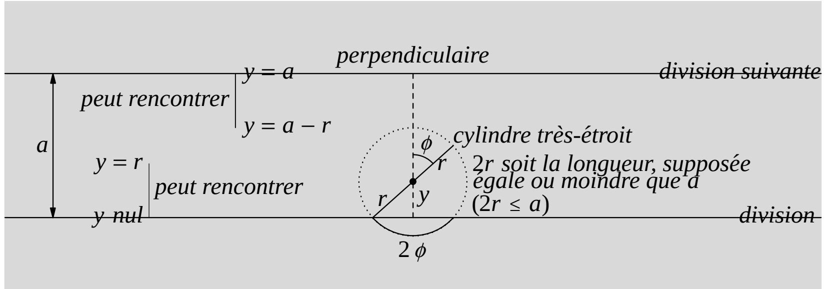
ainsi la somme dont il s'agit est $4r$. Depuis $y = a - r$ jusqu'à $y = a$, le cylindre peut rencontrer la division suivante, et il est visible que la somme de toutes les parties relatives à cette rencontre, est encore $4r$; $8r$ est donc la somme des toutes les parties relatives à la rencontre de l'une ou l'autre des divisions par le cylindre, dans le mouvement de son centre le long de la perpendiculaire. Mais le nombre de tous les arcs qu'il décrit en tournant en entier sur lui-même, à chaque point de cette perpendiculaire, est $2a\pi$; c'est le nombre de toutes les combinaisons possibles; la probabilité de la rencontre d'une des divisions du plan par le cylindre, est donc $\frac{4r}{a\pi}$. Si l'on projette un grand nombre de fois ce cylindre, le rapport du nombre de fois où le cylindre rencontrera l'une des divisions du plan, au nombre total des projections, sera par le n^o 16, à très-peu près, la valeur de $\frac{4r}{a\pi}$, ce qui fera connaître la valeur de la circonference 2π. On aura, par le même numéro, la probabilité que l'erreur de cette valeur sera comprise dans des limites données; et il est facile de voir que le rapport $\frac{2r}{a}$ qui, pour un nombre donné de projections, rend l'erreur à craindre la plus petite, est l'unité; ce qui donne la longueur du cylindre égale à l'intervalle des divisions."

Diagram illustrating Laplace's working



So much for Laplace's "je vais exposer en peu de mots" (*I shall explain it in a few words*)! Despite (or because of) all these words, Laplace's explanation of his working is not particularly explcit. Here is a more explicit summary (and, below, "needle" replaces "cylinder").

The angle ϕ is determined by, therefore a function of, the distance y of centre of the needle from the base division. It varies from $\phi = 0$ ($y = r$, half the length of the needle; or $y = a - r$) to $\phi = \frac{\pi}{2}$ ($y = 0$; or $y = a$). The angle $2\,\phi$ swept out for a given y is counted for each end, hence $4\,\phi$ for a given y. Hence the first (indefinite) integral $4\int \phi\, dy$ for the totality of (y, ϕ)-space such that the needle intersects a division. While ϕ is an awkward function of y, $y = r.\cos\phi$ is simple. Integrating by parts: $4\int \phi\, dy \;=\; 4\phi y - 4\int y\, d\phi \;=\; 4\phi y - 4r.\sin\phi + \text{constant}$. It remains to evaluate the constant—but this again is wordy! A better overall derivation is given on the next page.



Now disregarding intersection, let ϕ denote the angle between the needle and the perpendicular when the needle is thrown randomly onto the plane; then the position y of its centre has a uniform distribution on $(0, a)$, and ϕ a uniform distribution on $(-\frac{\pi}{2}, \frac{\pi}{2})$; ϕ, y are independent. Then every differential box $dy\, d\phi$ has the same probability. The total content of (y, ϕ)-space is $\pi \times a$, so the probability of $dy\, d\phi$ is $\dfrac{dy\, d\phi}{\pi\, a}$, since now $\displaystyle\int_{\phi=-\pi/2}^{\pi/2} \int_{y=0}^{a} \frac{dy\, d\phi}{\pi\, a} = 1$. Next, we find the probability of the subset of (y, ϕ)-space such that the needle intersects one or other of the *division* lines (it cannot intersect both, since $2\,r < a$).

For a given value of ϕ in $(-\frac{\pi}{2}, \frac{\pi}{2})$, y can range over $(0, r\cos\phi)$ or over $(a - r\cos\phi, a)$. These intervals do not overlap. Hence the probability of intersection is

$$P = \int_{\phi=-\pi/2}^{\pi/2} \int_{y=0}^{r\cos\phi} \frac{dy\, d\phi}{\pi\, a} + \int_{\phi=-\pi/2}^{\pi/2} \int_{y=a-r\cos\phi}^{a} \frac{dy\, d\phi}{\pi\, a}$$

$$= 2\int_{\phi=-\pi/2}^{\pi/2} \int_{y=0}^{r\cos\phi} \frac{dy\, d\phi}{\pi\, a} = 2\int_{-\pi/2}^{\pi/2} \frac{r\cos\phi}{\pi\, a}\, d\phi$$

$$= \frac{4\,r}{\pi\, a} = \frac{2\,L}{\pi\, a} \qquad [\, L \equiv 2\,a = \text{length of needle} \,] \quad [\bullet\bullet\bullet]$$

(agreeing with Laplace's result). Hence ... at last ... !!! :

$$\pi = \frac{4r}{a} \times \frac{1}{P} = \frac{2\,L}{a} \times \frac{1}{P} \qquad [❤]$$

Thus throwing a needle of length $L = 2\,r$ randomly onto a plane ruled with parallel lines a distance $a > 2\,r$ apart, many times, and counting the proportion P of throws in which the needle intersects a line, gives an empirical estimate of the value of π, using equation [❤].

But there is a simpler approach ... **!!**

This starts with the concept of the Expectation of a random variable. If a random variable X takes values $x_1, x_2, \ldots, x_k$ with probabilities $p_1, p_2, \ldots, p_k$ then the Expectation $\mathrm{E}(X)$ of X is defined as

$$\mathrm{E}(X) = p_1\, x_1 + p_2\, x_2 + \cdots + p_k\, x_k$$

Given N jointly distributed random variables $X_1, X_2, \ldots, X_N$ it is a basic fact (from one's first course on Probability) that the Expectation of their sum is the sum of their Expectations:

$$\mathrm{E}(X_1 + X_2 + \cdots + X_N) = \mathrm{E}(X_1) + \mathrm{E}(X_2) + \cdots + \mathrm{E}(X_N)$$

If a random variable X takes only two values: 0 with probability p_0, 1 with probability p_1, then the Expectation of X is

$$\mathrm{E}(X) = 0 \times p_0 + 1 \times p_1 = p_1 = \mathrm{P}(X = 1)$$

the probability that $X = 1$.

Consider a planar 'quasi-curve' made up of a number N of very short straight-line segments, each of length, say, δL. Consider throwing one of these segments onto the ruled plane at random, as above. Let this segment have probability δP of intersecting a division line. Score $X = 1$ if it does, $X = 0$ if it does not. Then $\mathrm{E}(X) = \delta P$, as above. For each segment $1, 2, \ldots, N$, score $X_1, X_2, \ldots, X_N$ correspondingly when the whole 'quasi-curve' is thrown down at random. Then the total number I of intersections of the 'curve' with the division lines is $I = X_1 + X_2 + \cdots + X_N$ whose Expectation is

$$\mathrm{E}(I) = \mathrm{E}(\textstyle\sum X_i) = \sum \mathrm{E}(X_i) = N\delta P$$

since the segments are identical. Now, for segments of given equal length, the length of the 'curve' is proportional to N. Hence, for any such 'curve', $\mathrm{E}(I) \propto L$: the expected number of intersections is proportional to its length, regardless of its shape. Hence, for some constant C, $\mathrm{E}(I) = C \times L$.

Any smooth curve can be arbitrarily approximated by sufficiently many sufficiently short segments, so again $\mathrm{E}(I) = C \times L$. If the curve is a circle of diameter $2\,r = a$, the distance between divisions, then the number of intersections is always 2, so $\mathrm{E}(I) = 2$, $L = \pi\, a$, so $2 = C\,\pi\, a$, so $C = 2/(\pi\, a)$ and hence $\mathrm{E}(I) = 2L/(\pi\, a)$, which is Laplace's/Buffon's result at [•••] above. **QED**

But now it gets wiggly ⟶

Buffon's Thread

Since Buffon had a needle, he must have had a thread — else what was he doing with that needle? (Apart from constantly dropping it, and having to fumble for it on the floorboards — which is probably what gave him the idea for his Needle Problem: as an obsessively observant Natural Historian, he no doubt compulsively recorded whether or not he found it lying across the line between two boards). But no-one has yet spoken about Buffon's Thread. Now is the time.

As established on the previous page, if a curve of length L is dropped randomly onto a plane ruled with parallel lines distant a apart, then the Expectation of the number I of intersections of the curve with the lines is

$$\mathrm{E}\,(I\,) \;=\; \frac{2\,L}{\pi\,a} \qquad [\,✌\,]$$

regardless of the shape of the curve. So the curve can have a different shape every time it is dropped, so long as it remains of the same length L each time. and the result [✌] will remain true.

Therefore the curve can be a flexible thread, of length L, adopting whatever shape the hazards of dynamics impose on it when it hits the board. All you need to do is to count the number of intersections of the thread with the lines, once it is lying on the board. And that is the Buffon Thread Problem — already solved at [✌].

Finally, as a send-off . . .

As noted above, because there was Buffon's Needle, there must have been Buffon's Thread — and this had non-trivial consequences.

We have all heard of Occam's Razor:

Numquam ponenda est pluralitas sine necessitate **a**
Entia non sunt multiplicanda praeter necessitatem **b**

(*don't dream up anything that you don't absolutely have to*)

But if Occam had a razor, then he must have had a beard — else why (by his own principles) would he have to have a razor?

So Herewith **Occam's Beard**

Anything that can be imagined as possible must somewhere be true

(I appeal for an elegant translation)

There are examples in many sciences — the Higgs Boson finally came to light (as it were . . .).

Especially in deep-sea Marine Biology, there are many phenomena which were discovered by crawling through Occam's Beard.

a As in Occam's own writings: *Quaestiones et decisiones in quattuor libros Sententiarum Petri Lombardi*.

b Attributed to Occam, but reportedly not found in his extant writings.