Linear elasticity

Equation of motion

Notation

Cauchy stress tensor σ:

$$\sigma = \begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{bmatrix} = \begin{bmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{yx} & \sigma_{yy} & \sigma_{yz} \\ \sigma_{zx} & \sigma_{zy} & \sigma_{zz} \end{bmatrix} = \begin{bmatrix} \sigma_{x} & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_{y} & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_{z} \end{bmatrix}$$

Mass density ρ, Body force vector $\mathbf{g}$, Displacement vector $\mathbf{u}$, Time t

Equation (dynamic problem)

$$\boldsymbol{\nabla} \cdot \sigma^{\mathrm{T}} + \rho \mathbf{g} = \rho \frac{\partial^2 \mathbf{u}}{\partial t^2}$$

Equation (static problem)

$$\boldsymbol{\nabla} \cdot \sigma^{\mathrm{T}} + \rho \mathbf{g} = \mathbf{0}$$

Confirmation of the expression

According to the cheet sheet,

$$(\boldsymbol{\nabla} \cdot \sigma^{\mathrm{T}})_i = \partial_j \sigma_{ij} = \frac{\partial \sigma_{ij}}{\partial x_j}$$

Therefore,

$$\frac{\partial \sigma_{ij}}{\partial x_j} + \rho g_i = 0$$

This is correct.

Strain-displacement equation

Notation

Strain tensor ε:

$$\varepsilon = \begin{bmatrix} \varepsilon_{11} & \varepsilon_{12} & \varepsilon_{13} \\ \varepsilon_{21} & \varepsilon_{22} & \varepsilon_{23} \\ \varepsilon_{31} & \varepsilon_{32} & \varepsilon_{33} \end{bmatrix} = \begin{bmatrix} \varepsilon_{xx} & \varepsilon_{xy} & \varepsilon_{xz} \\ \varepsilon_{yx} & \varepsilon_{yy} & \varepsilon_{yz} \\ \varepsilon_{zx} & \varepsilon_{zy} & \varepsilon_{zz} \end{bmatrix} = \begin{bmatrix} \varepsilon_{x} & \gamma_{xy}/2 & \gamma_{xz}/2 \\ \gamma_{yx}/2 & \varepsilon_{y} & \gamma_{yz}/2 \\ \gamma_{zx}/2 & \gamma_{zy}/2 & \varepsilon_{z} \end{bmatrix}$$

Equation

$$\varepsilon = \frac{1}{2}(\boldsymbol{\nabla} \otimes \mathbf{u} + (\boldsymbol{\nabla} \otimes \mathbf{u})^{\mathrm{T}})$$

Confirmation of the expression

According to the cheet sheet,

$$(\boldsymbol{\nabla} \otimes \mathbf{u})_{ij} = \partial_i u_j = \frac{\partial u_j}{\partial x_i}$$

$$((\boldsymbol{\nabla} \otimes \mathbf{u})^{\mathrm{T}})_{ij} = \partial_j u_i = \frac{\partial u_i}{\partial x_j}$$

Therefore,

$$\varepsilon_{ij} = \frac{1}{2}(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i})$$

This is correct.

Constitutive equation

Notation

Kronecker delta δ:

$$\delta_{ij} = \begin{cases} 0 & \text{if } i \neq j \\ 1 & \text{if } i = j \end{cases}$$

or

$$\delta_{ij} = \mathrm{I}_{ij} \quad \text{for} \quad \mathrm{I} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Equation (with Lamé parameters λ, μ)

$$\sigma = \lambda(\boldsymbol{\nabla} \cdot \mathbf{u})\mathrm{I} + \mu(\boldsymbol{\nabla} \otimes \mathbf{u} + (\boldsymbol{\nabla} \otimes \mathbf{u})^{\mathrm{T}})$$

Confirmation of the expression

According to the cheet sheet,

$$\boldsymbol{\nabla} \cdot \mathbf{u} = \partial_i u_i = \frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2} + \frac{\partial u_3}{\partial x_3}$$

$$(\boldsymbol{\nabla} \otimes \mathbf{u})_{ij} = \partial_i u_j = \frac{\partial u_j}{\partial x_i}$$

$$((\boldsymbol{\nabla} \otimes \mathbf{u})^{\mathrm{T}})_{ij} = \partial_j u_i = \frac{\partial u_j}{\partial x_i}$$

Therefore,

$$\sigma_{ij} = \lambda(\frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2} + \frac{\partial u_3}{\partial x_3})\delta_{ij} + \mu(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}) = \lambda \varepsilon_{kk} \delta_{ij} + 2\mu \varepsilon_{ij}$$

This is correct.

About σ

Notation

$$\partial_i^2 = \frac{\mathrm{d}^2}{\mathrm{d}x_i^2}$$

$$\Delta = \boldsymbol{\nabla} \cdot \boldsymbol{\nabla} = \boldsymbol{\nabla}^{\mathrm{T}} \boldsymbol{\nabla} = \partial_1^2 + \partial_2^2 + \partial_3^2$$

Equations

$$\boldsymbol{\nabla} \cdot \sigma^{\mathrm{T}} + \rho \mathbf{g} = \mathbf{0}$$

$$\sigma = \lambda(\boldsymbol{\nabla} \cdot \mathbf{u})\mathrm{I} + \mu(\boldsymbol{\nabla} \otimes \mathbf{u} + (\boldsymbol{\nabla} \otimes \mathbf{u})^{\mathrm{T}})$$

Expansion

Matrix form

$$\sigma = \lambda \boldsymbol{\nabla}^{\mathrm{T}} \mathbf{u} \mathrm{I} + \mu(\boldsymbol{\nabla} \mathbf{u}^{\mathrm{T}} + (\boldsymbol{\nabla} \mathbf{u}^{\mathrm{T}})^{\mathrm{T}})$$

$$\sigma^{\mathrm{T}} = \lambda \mathrm{I}^{\mathrm{T}} (\boldsymbol{\nabla}^{\mathrm{T}} \mathbf{u})^{\mathrm{T}} + \mu((\boldsymbol{\nabla} \mathbf{u}^{\mathrm{T}})^{\mathrm{T}} + \boldsymbol{\nabla} \mathbf{u}^{\mathrm{T}}) = \sigma$$

$$
\begin{aligned}
\nabla \cdot \sigma^{\mathrm{T}} &= \boldsymbol{\nabla}^{\mathrm{T}} \sigma^{\mathrm{T}} \\
&= \boldsymbol{\nabla}^{\mathrm{T}} \sigma \\
&= \lambda \boldsymbol{\nabla}^{\mathrm{T}} \boldsymbol{\nabla}^{\mathrm{T}} \mathbf{u} \mathrm{I} + \mu (\boldsymbol{\nabla}^{\mathrm{T}} \boldsymbol{\nabla} \mathbf{u}^{\mathrm{T}} + \boldsymbol{\nabla}^{\mathrm{T}} (\boldsymbol{\nabla} \mathbf{u}^{\mathrm{T}})^{\mathrm{T}}) \\
&= \lambda \boldsymbol{\nabla}^{\mathrm{T}} \boldsymbol{\nabla}^{\mathrm{T}} \mathbf{u} + \mu (\Delta \mathbf{u}^{\mathrm{T}} + \boldsymbol{\nabla}^{\mathrm{T}} (\boldsymbol{\nabla} \mathbf{u}^{\mathrm{T}})^{\mathrm{T}})
\end{aligned}
$$

component form

$\sigma_{ij} = \lambda \partial_k u_k \delta_{ij} + \mu (\partial_i u_j + \partial_j u_i)$

$(\sigma^{\mathrm{T}})_{ij} = \lambda \partial_k u_k \delta_{ji} + \mu (\partial_j u_i + \partial_i u_j) = \sigma_{ij}$

$$
\begin{aligned}
(\boldsymbol{\nabla} \cdot \sigma^{\mathrm{T}})_i &= \partial_j \sigma_{ij} \\
&= \lambda \partial_j \partial_k u_k \delta_{ij} + \mu (\partial_j \partial_j u_i + \partial_j \partial_i u_j) \\
&= \lambda \partial_i \partial_k u_k + \mu (\partial_j^2 u_i + \partial_i \partial_j u_j)
\end{aligned}
$$

For FEM

Strong form

$\boldsymbol{\nabla} \cdot \sigma^{\mathrm{T}} + \rho \mathbf{g} = \mathbf{0}$

or

$\frac{\partial \sigma_{ij}}{\partial x_j} + \rho g_i = 0$

To weak form

Using test function vectors $\hat{\mathbf{u}}$ in a volume V,

$\int_V \frac{\partial \sigma_{ij}}{\partial x_j} \hat{u}_i \mathrm{d}V + \int_V \rho g_i \hat{u}_i \mathrm{d}V = 0$

Integrating by parts,

$\int_V \frac{\partial \sigma_{ij}}{\partial x_j} \hat{u}_i \mathrm{d}V = \int_V \frac{\partial}{\partial x_j} (\sigma_{ij} \hat{u}_i) \mathrm{d}V - \int_V \sigma_{ij} \frac{\partial \hat{u}_i}{\partial x_j} \mathrm{d}V$

From Gauss's divergence theorem, using vectors $\mathbf{n}$ normal to S which is the surface of V,

$\int_V \frac{\partial \sigma_{ij}}{\partial x_j} \hat{u}_i \mathrm{d}V = \int_S n_j \sigma_{ij} \hat{u}_i \mathrm{d}S - \int_V \sigma_{ij} \frac{\partial \hat{u}_i}{\partial x_j} \mathrm{d}V$

From Cauchys stress theorem, stress vectors $\mathbf{t}$ are

$t_i = \sigma_{ij} n_j$

Therefore, we obtain

$\int_V \sigma_{ij} \frac{\partial \hat{u}_i}{\partial x_j} \mathrm{d}V = \int_S t_i \hat{u}_i \mathrm{d}S + \int_V \rho g_i \hat{u}_i \mathrm{d}V$

For GetFEM++

The term

$\int_V \sigma_{ij} \frac{\partial \hat{u}_i}{\partial x_j} \mathrm{d}V \rightarrow$ "(lambda * Trace(Grad_u) * Id(qdim(u)) + mu * (Grad_u + Grad_u')) : Grad_Test_u"

because,

$\sigma_{ij} \rightarrow \sigma_{ij} = \lambda \partial_k u_k \delta_{ij} + \mu(\partial_i u_j + \partial_j u_i) \rightarrow$ "(lambda * Trace(Grad_u) * Id(qdim(u)) + mu * (Grad_u + Grad_u'))

$\frac{\partial \hat{u}_i}{\partial x_j} \rightarrow$ "Grad_Test_u"