Neo-Hookean material laws

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1 Kinematics

Deformation gradient:

$$\boldsymbol{F} = I + \nabla \boldsymbol{u} \tag{1}$$

Right Cauchy-Green strain tensor:

$$\boldsymbol{C} = \boldsymbol{F}^T \boldsymbol{F} \tag{2}$$

Left Cauchy-Green strain tensor:

$$\boldsymbol{b} = \boldsymbol{F}\boldsymbol{F}^T \tag{3}$$

First invariant (trace) of $\boldsymbol{C}$ and $\boldsymbol{b}$:

$$I_1(\boldsymbol{C}) = I_1(\boldsymbol{b}) = \|\boldsymbol{F}\|^2 \tag{4}$$

Third invariant (determinant) of $\boldsymbol{C}$ and $\boldsymbol{b}$:

$$I_3(\boldsymbol{C}) = I_3(\boldsymbol{b}) = |\boldsymbol{F}|^2 \tag{5}$$

Isochoric part of the right Cauchy-Green strain tensor:

$$\hat{\boldsymbol{C}} = |\boldsymbol{F}|^{-2/3}\, \boldsymbol{F}^T \boldsymbol{F} \tag{6}$$

Isochoric part of the left Cauchy-Green strain tensor:

$$\hat{\boldsymbol{b}} = |\boldsymbol{F}|^{-2/3}\, \boldsymbol{F}\boldsymbol{F}^T \tag{7}$$

First invariant (trace) of $\hat{\boldsymbol{C}}$ and $\hat{\boldsymbol{b}}$:

$$I_1(\hat{\boldsymbol{C}}) = I_1(\hat{\boldsymbol{b}}) = |\boldsymbol{F}|^{-2/3}\, \|\boldsymbol{F}\|^2 \tag{8}$$

Third invariant (determinant) of $\hat{\boldsymbol{C}}$ and $\hat{\boldsymbol{b}}$:

$$I_3(\hat{\boldsymbol{C}}) = I_3(\hat{\boldsymbol{b}}) = 1 \tag{9}$$

Logarithmic strain tensors:

$$\ln \boldsymbol{b} = \ln \boldsymbol{C} = \ln\left(\boldsymbol{F}\boldsymbol{F}^T\right) = \ln\left(\boldsymbol{F}^T\boldsymbol{F}\right) = \ln \boldsymbol{F} + \ln \boldsymbol{F}^T \tag{10}$$

Isochoric logarithmic strain tensors:

$$\ln \hat{\boldsymbol{b}} = \ln \hat{\boldsymbol{C}} = \ln\left(|\boldsymbol{F}|^{-2/3}\, \boldsymbol{F}\boldsymbol{F}^T\right) = \ln\left(|\boldsymbol{F}|^{-2/3}\, \boldsymbol{F}^T\boldsymbol{F}\right) = \operatorname{dev}(\ln \boldsymbol{b}) = \operatorname{dev}(\ln \boldsymbol{C}) \tag{11}$$

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2 Constitutive laws

Lamé's first parameter λ.
Lamé's second parameter μ (also commonly referred to as shear modulus G).
Bulk modulus κ:

$$\kappa = \lambda + \frac{2\mu}{3} \tag{12}$$

Strain energy density function from (Bonet and Wood, 1997, Eq. 5.27):

$$W_1 = \frac{\lambda}{2}(\ln|\boldsymbol{F}|)^2 - \mu \ln|\boldsymbol{F}| + \frac{\mu}{2}\left(\|\boldsymbol{F}\|^2 - 3\right) \tag{13}$$

Corresponding Cauchy stress from (Bonet and Wood, 1997, Eq. 5.29):

$$\boldsymbol{\sigma}_1 = \frac{\lambda \ln|\boldsymbol{F}|\boldsymbol{I} + \mu\left(\boldsymbol{F}\boldsymbol{F}^T - \boldsymbol{I}\right)}{|\boldsymbol{F}|} \tag{14}$$

The strain energy according to Eq. (13) is also used in (Simo and Ortiz, 1985), cf. Eqs. 2.5 and 2.6.
Strain energy density function from (Wriggers, 2008, Eqs. 3.116 and 3.118):

$$W_2 = \frac{\lambda}{4}\left(|\boldsymbol{F}|^2 - 1\right) - \frac{\lambda}{2}\ln|\boldsymbol{F}| - \mu\ln|\boldsymbol{F}| + \frac{\mu}{2}\left(\|\boldsymbol{F}\|^2 - 3\right) \tag{15}$$

Corresponding Cauchy stress from (Wriggers, 2008, Eq. 3.120):

$$\boldsymbol{\sigma}_2 = \frac{\lambda\frac{|\boldsymbol{F}|^2 - 1}{2}\boldsymbol{I} + \mu\left(\boldsymbol{F}\boldsymbol{F}^T - \boldsymbol{I}\right)}{|\boldsymbol{F}|} \tag{16}$$

Strain energy density function from (Simo et al., 1985, Eq. 4.4):

$$W_3 = \frac{\kappa}{2}(\ln|\boldsymbol{F}|)^2 + \frac{\mu}{2}\left(|\boldsymbol{F}|^{-2/3}\,\|\boldsymbol{F}\|^2 - 3\right) \tag{17}$$

Corresponding Cauchy stress from (Simo et al., 1985, Eq. 4.5):

$$\boldsymbol{\sigma}_3 = \frac{\kappa \ln|\boldsymbol{F}|\boldsymbol{I} + \mu\,|\boldsymbol{F}|^{-2/3}\,\mathrm{dev}\left(\boldsymbol{F}\boldsymbol{F}^T\right)}{|\boldsymbol{F}|} \tag{18}$$

Strain energy density function from (Wriggers, 2008, Eq. 3.122):

$$W_4 = \frac{\kappa}{4}\left(|\boldsymbol{F}|^2 - 1\right) - \frac{\kappa}{2}\ln|\boldsymbol{F}| + \frac{\mu}{2}\left(|\boldsymbol{F}|^{-2/3}\,\|\boldsymbol{F}\|^2 - 3\right) \tag{19}$$

Corresponding Cauchy stress based on Eq. 3.127 or Eq. 3.274 from (Wriggers, 2008):

$$\boldsymbol{\sigma}_4 = \frac{\kappa\frac{|\boldsymbol{F}|^2 - 1}{2}\boldsymbol{I} + \mu\,|\boldsymbol{F}|^{-2/3}\,\mathrm{dev}\left(\boldsymbol{F}\boldsymbol{F}^T\right)}{|\boldsymbol{F}|} \tag{20}$$

The strain energy according to Eq. (19) is similar to the one defined in (Simo and Miehe, 1992), cf. Eq. 2.37. However, in (Simo and Miehe, 1992) a factor of $\frac{1}{2}$ is missing in front of κ.
Strain energy density function from (Hencky, 1931, 1933) reproduced in (Anand, 1986, Eq. 32):

$$W_5 = \frac{\lambda}{2}(\ln|\boldsymbol{F}|)^2 + \frac{\mu}{4}\left\|\ln\left(\boldsymbol{F}\boldsymbol{F}^T\right)\right\|^2 \tag{21}$$

In this equation the factor of $\frac{1}{2}$ applied to λ is missing in the original paper (Anand, 1986, Eq. 32), but it is available e.g. in (Gilchrist et al. 2012).
Corresponding Cauchy stress from (Anand, 1986, Eq. 33):

$$\boldsymbol{\sigma}_5 = \frac{\lambda \ln|\boldsymbol{F}|\boldsymbol{I} + \mu \ln\left(\boldsymbol{F}\boldsymbol{F}^T\right)}{|\boldsymbol{F}|} \tag{22}$$

Strain energy density function from (Neff et al., 2015, Eq. 1.5):

$$W_6 = \frac{\kappa}{2}(\ln|\boldsymbol{F}|)^2 + \frac{\mu}{4}\left\|\mathrm{dev}\left(\ln\left(\boldsymbol{F}\boldsymbol{F}^T\right)\right)\right\|^2 \tag{23}$$

Corresponding Cauchy stress from (Neff et al., 2015, Eq. 1.5):

$$\boldsymbol{\sigma}_6 = \frac{\kappa \ln|\boldsymbol{F}|\boldsymbol{I} + \mu\,\mathrm{dev}\left(\ln\left(\boldsymbol{F}\boldsymbol{F}^T\right)\right)}{|\boldsymbol{F}|} \tag{24}$$